

UPDATED $B_q \rightarrow \bar{\ell}\ell$ IN THE STANDARD MODEL AT HIGHER ORDERS

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The latest standard model predictions of rare leptonic $B_q \rightarrow \bar{\ell}\ell$ decays are reviewed including recent results of next-to-leading electroweak and next-to-next-to-leading QCD corrections.

1 Introduction

The rare leptonic $B_q \rightarrow \bar{\ell}\ell$ decays of neutral B -mesons – with $q = d, s$ and $\ell = e, \mu, \tau$ – provide interesting tests of flavor-changing neutral-current (FCNC) transitions mediated by $b \rightarrow q \bar{\ell}\ell$ at the parton level as predicted by the standard model (SM). Besides the loop-suppression of FCNC decays in the SM, the SM contribution is in addition helicity suppressed, which leads the very rare branching ratios $\sim \mathcal{O}(10^{-9} - 10^{-10})$ for $\ell = \mu$, and offers tests of non-standard effects due to scalar and pseudo-scalar interactions.

Utilising their full data sets of 3fb^{-1} and 25fb^{-1} by the year 2013, the two experiments LHCb¹ and CMS² succeeded to observe the CP-averaged time-integrated branching ratio³ of the $B_s \rightarrow \bar{\mu}\mu$ channel with a statistical significance of 4.0σ and 4.3σ , respectively. The evidence of the measurement of the other muonic channel, $B_d \rightarrow \bar{\mu}\mu$, is about 2.0σ in both experiments and correlated with $B_s \rightarrow \bar{\mu}\mu$. The combination of the LHCb and CMS measurements⁴ yields

$$\begin{aligned}\bar{\mathcal{B}}[B_s \rightarrow \bar{\mu}\mu]_{\text{exp.}} &= (2.9 \pm 0.7) \times 10^{-9}, \\ \bar{\mathcal{B}}[B_d \rightarrow \bar{\mu}\mu]_{\text{exp.}} &= (3.6^{+1.6}_{-1.4}) \times 10^{-10},\end{aligned}\tag{1}$$

where the uncertainties are currently dominated by the statistical error, but include systematic sources, as well. As a consequence of the combination of LHCb and CMS measurements one can consider the decay $B_s \rightarrow \bar{\mu}\mu$ as observed (i.e. $> 5\sigma$) while the yield of $B_d \rightarrow \bar{\mu}\mu$ is not statistically significant yet (i.e. $< 3\sigma$).

The future experimental prospects are rather promising for $B_s \rightarrow \bar{\mu}\mu$. For example LHCb is expected⁵ to reach an accuracy of 0.5×10^{-9} after Run 2 of LHC in the years 2015-2017 and even 0.15×10^{-9} after the detector upgrade with a data set of 50fb^{-1} . The latter absolute error corresponds to about 5 % relative error given the current central value of the measured branching ratio. Further, the large data set allows to measure the ratio $\mathcal{R}_\mu \equiv \bar{\mathcal{B}}[B_d \rightarrow \bar{\mu}\mu]/\bar{\mathcal{B}}[B_s \rightarrow \bar{\mu}\mu]$ with a relative error of 35 %. A recent comprehensive study⁶ of the reach of CMS expects that with the current detector Run 2 of LHC will deliver about 100fb^{-1} , which enables CMS to measure $\bar{\mathcal{B}}[B_s \rightarrow \bar{\mu}\mu]$ with a significance $> 10\sigma$ and a relative error of 15 % and for \mathcal{R}_μ about 70 %. After Run 3 of LHC in the years 2019-2021, these relative errors become 12 % and 50 %, respectively, based on a data set of about 300fb^{-1} , and the statistical significance for the decay $B_d \rightarrow \bar{\mu}\mu$ might reach the 3σ level. In the far future, the potential luminosity upgrade of the LHC

machine to HL-LHC (starting with the year 2023) will deliver about 3000 fb^{-1} , however, without substantial reduction of currently assumed systematic errors, the relative error of $\overline{\mathcal{B}}[B_s \rightarrow \bar{\mu}\mu]$ will remain around 12 %, whereas $\overline{\mathcal{B}}[B_d \rightarrow \bar{\mu}\mu]$ can be discovered (i.e. $> 5\sigma$) and a relative error of 18 % can be reached. The latter would imply about 21 % relative error in \mathcal{R}_μ .

In view of future experimental uncertainties below 10 %, the theoretical predictions have been updated in several respects over the last two years. These efforts were further motivated by the steady progress of lattice determinations of the B_q -decay constant that constituted in the past the major source of uncertainty and has been reduced nowadays to the level of other parametric uncertainties. First, we will give a short introduction of the theoretical treatment of $B_q \rightarrow \bar{\ell}\ell$ decays including higher order electroweak (EW) and QCD corrections in section 2. Then we will discuss in section 3 the latest SM predictions and present a detailed uncertainty budget, which can bear comparison with experimental prospects.

2 Theory of $B_q \rightarrow \bar{\ell}\ell$

Rare $\Delta B = 1$ FCNC decays can be conveniently described by an effective theory of the electroweak interactions after the application of an operator product expansion (OPE). The OPE corresponds to an expansion in external momenta of the considered processes that are of the order of the bottom-quark mass, $m_b \sim 5 \text{ GeV}$, which are much smaller than the internal heavy W -boson mass $m_W \sim 80 \text{ GeV}$. The effective Lagrangian takes a systematic expansion

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{QCD} \times \text{QED}}(u, d, s, c, b, e, \mu, \tau) + \mathcal{L}_{\text{dim}=6} + \mathcal{L}_{\text{dim}=8} + \dots \quad (2)$$

where the first term describes the QCD and QED gauge interactions of $\text{dim} = 4$ of the light ($N_f = 5$) quarks and leptons. The second term represents the leading effect of flavor-changing operators O_i of $\text{dim} = 6$ that mediate $|\Delta B| = |\Delta Q| = 1$ ($Q = D, S$) processes

$$\mathcal{L}_{\text{dim}=6} = \mathcal{N}_{\text{eff}}^q \sum_i C_i(\mu_b) O_i + \text{h.c.}, \quad \mathcal{N}_{\text{eff}}^q = \frac{G_F^2 m_W^2}{\pi^2} V_{tb} V_{tq}^*, \quad (3)$$

whereas the higher dimensional operators ($\text{dim} > 6$) are suppressed by $(m_b/m_W)^2 \sim 0.3\%$. Each operator has an effective coupling C_i (Wilson coefficient), which is determined in the matching at a so-called matching scale of order m_W and has been evolved with the aid of the renormalization group (RG) equation of the effective theory to the low-energy scale $\mu_b \sim m_b$. This procedure provides a systematic framework to account for the large logarithmic contributions $\sim \alpha_s^n \ln^n(m_W/\mu_b)$ from radiative corrections to all orders in the QCD coupling α_s . Nowadays, the $C_i(\mu_b)$ relevant for $b \rightarrow q \bar{\ell}\ell$ transitions are known to rather high orders including the RG evolution^{7,8}. The choice of the normalization factor⁹ $\mathcal{N}_{\text{eff}}^q$ provides better convergence properties with regard to higher order EW corrections. It contains besides Fermi's constant, G_F , the product of elements of the Cabibbo-Kobayashi-Maskawa (CKM) quark-mixing matrix.

Concerning $B_q \rightarrow \bar{\ell}\ell$, at leading order (LO) in EW/QED interactions, one single operator

$$O_{10} = (\bar{q}\gamma_\mu P_L b)(\bar{\ell}\gamma^\mu \gamma_5 \ell), \quad P_L = \frac{1 - \gamma_5}{2} \quad (4)$$

is relevant. Since O_{10} is a conserved current under QCD to this order in EW/QED interactions, its Wilson coefficient^a does not evolve and does not depend on μ_b . The LO contribution¹⁰ of C_{10} has as strong dependence on the renormalization scheme of the top-quark mass, which is cancelled by the next-to-leading^{11,12,13,14} (NLO) and next-to-next-to-leading¹⁵ (NNLO) order QCD corrections. The inclusion of NNLO corrections reduce the related uncertainties from 1.8 % at NLO to less than 0.2 % at the level of the branching ratio¹⁵.

^aHere we use the convention $C_{10} = -2C_A$ compared to^{15,23} and $C_{10} = \tilde{c}_{10}$ to¹⁸. It differs by a factor of sine squared of the weak mixing angle to c_{10} of^{7,8}: $C_{10} = s_W^2 c_{10}$ at LO in EW interactions.

With this remarkable control of short-distance QCD corrections, also higher order EW/QED corrections have to be considered. NLO EW corrections to the matching of C_{10} had been derived in the large top-quark mass limit¹⁶ and used to point out that EW renormalization scheme dependences¹⁷ of the branching ratio are a sizeable source of $\sim 7\%$ to 5% uncertainty. Recently, the complete NLO EW matching corrections to C_{10} have been calculated¹⁸ employing several different choices of renormalization schemes of the relevant parameters. The best convergence properties, i.e., small NLO corrections compared to the LO contribution, were obtained in the scheme that eliminates the ratio α_e/s_W^2 in favor of G_F – as done with the choice of $\mathcal{N}_{\text{eff}}^q$ – and in the scheme where both quantities entering the ratio α_e/s_W^2 are renormalized in the $\overline{\text{MS}}$ scheme¹⁹. Due to the removal of EW scheme dependences at NLO, the previous $\sim 7\%$ uncertainty of the branching ratio at LO was reduced to about 0.6% at NLO. Furthermore, at NLO in QED, operator mixing of O_{10} with other operators in (3) leads to a non-trivial evolution of C_{10} and gives rise to μ_b -dependence. The necessary ingredients for a consistent RG evolution had been provided before^{7,8}, with explicit results given in¹⁸.

At LO in QED, the matrix element of $B_q \rightarrow \bar{\ell}\ell$ is easily derived from (3) by restricting the sum to $i = 10$ and neglecting QED interactions below the scale μ_b , i.e., replacing $\mathcal{L}_{\text{QCD} \times \text{QED}} \rightarrow \mathcal{L}_{\text{QCD}}$

$$\frac{i\mathcal{M}_{\text{LO QED}}}{\mathcal{N}_{\text{eff}}^q C_{10}(\mu_b)} = \langle \bar{\ell}\ell | O_{10} | B_q \rangle = \langle \bar{\ell}\ell | \bar{\ell}\gamma^\mu \gamma_5 \ell | 0 \rangle \langle 0 | \bar{q}\gamma_\mu P_L b | B_q \rangle = m_\ell (\bar{u}_\ell \gamma_5 v_\ell) f_{B_q}. \quad (5)$$

This allows to factorize the currents present in O_{10} and the B_q -decay constant f_{B_q} is determined in lattice calculations²⁰. Here we include NLO EW matching corrections in C_{10} as they are complete and do not cause μ_b dependence. The latter is entirely due to photonic corrections within the effective theory and will be cancelled by NLO QED corrections to the matrix element. The scale variation of μ_b from $m_b/2$ to $2m_b$ leads to a variation of the branching ratio of 0.3% , which corresponds to a typical size of a $\mathcal{O}(\alpha_e)$ correction. The actual calculation of the lacking virtual NLO QED corrections to the matrix elements seems very challenging bearing in mind that QED corrections prevent a simple factorization of the matrix element as in (5) at LO.

The average time-integrated branching ratio of $B_q \rightarrow \bar{\ell}\ell$ can be obtained to a very high accuracy in the SM as $\overline{\mathcal{B}}[B_q \rightarrow \bar{\ell}\ell] = \Gamma[B_q \rightarrow \bar{\ell}\ell] / \Gamma_H^q$ where Γ_H^q denotes the heavier mass-eigenstate's total width³. With (3) and (5), it takes the form

$$\overline{\mathcal{B}}[B_q \rightarrow \bar{\ell}\ell] = \frac{|\mathcal{N}_{\text{eff}}^q|^2 m_{B_q}^3 f_{B_q}^2}{8\pi \Gamma_H^q} \left(\frac{m_\ell}{m_{B_q}} \right)^2 \sqrt{1 - \frac{4m_\ell^2}{m_{B_q}^2}} |C_{10}(\mu_b)|^2 + \mathcal{O}(\alpha_e) \quad (6)$$

exhibiting the helicity-suppression factor m_ℓ/m_{B_q} of the lepton and B_q -meson masses. The $\mathcal{O}(\alpha_e)$ term represents now the previously discussed virtual NLO QED corrections as well as soft photon bremsstrahlung. The latter might receive large enhancements in the presence of kinematical cuts in the experimental analysis, or initial-state radiation of photons, which may lift the helicity suppression. The initial-state radiation is infrared safe due to electrically neutral B_q meson and its interference with final-state radiation is helicity suppressed. Moreover, it is strongly phase space suppressed²¹ within the signal window in the dilepton invariant mass chosen by LHCb¹ and CMS², which allows to ignore it in the experimental analysis and discard it on the theoretical side by definition. Concerning photon bremsstrahlung from the final-state leptons, LHCb¹ and CMS² apply PHOTOS²² to extrapolate beyond cuts in the dilepton invariant mass, such that soft QED logarithms cancel out and the experimental quantity (1) is equivalent to (6) up to terms that do not receive an extra enhancement¹⁷.

3 Updated prediction of $B_q \rightarrow \bar{\ell}\ell$

The current theoretical status has been summarized in the preceding section. The novel calculations of NNLO QCD¹⁵ and NLO EW¹⁸ short-distance corrections to $C_{10}(\mu_b)$ remove important

Table 1: Relative uncertainties from various sources in $\overline{\mathcal{B}}[B_s \rightarrow \bar{\ell}\ell]$ and $\overline{\mathcal{B}}[B_d \rightarrow \bar{\ell}\ell]$. In the last column they are added in quadrature.

	f_{B_q}	CKM	τ_H^q	M_t	α_s	other param.	non-param.	Σ
$\overline{\mathcal{B}}[B_s \rightarrow \bar{\ell}\ell]$	4.0 %	4.3 %	1.3 %	1.6 %	0.1 %	< 0.1 %	1.5 %	6.4 %
$\overline{\mathcal{B}}[B_d \rightarrow \bar{\ell}\ell]$	4.5 %	6.9 %	0.5 %	1.6 %	0.1 %	< 0.1 %	1.5 %	8.5 %

renormalization scheme dependences at the matching scale present at lower orders. The prediction of branching ratios is formally not complete at the NLO in QED due to the lack of virtual corrections to the matrix element of (3). These corrections will remove scheme and the μ_b scale dependence still present in current predictions, however, they do not undergo extra enhancement, contrary to the included ones that receive either $1/s_W^2$ or $\ln^2(m_W/\mu_b)$ enhancement factors. In consequence, the residual μ_b dependence of our result is very weak, as discussed in the previous section.

Based on (5) and (6), most recent predictions for the branching ratios in the SM have been presented²³ for all decay channels

$$\begin{aligned}
\overline{\mathcal{B}}[B_s \rightarrow \bar{e}e] &= (8.54 \pm 0.55) \times 10^{-14}, & \overline{\mathcal{B}}[B_d \rightarrow \bar{e}e] &= (2.48 \pm 0.21) \times 10^{-15}, \\
\overline{\mathcal{B}}[B_s \rightarrow \bar{\mu}\mu] &= (3.65 \pm 0.23) \times 10^{-9}, & \overline{\mathcal{B}}[B_d \rightarrow \bar{\mu}\mu] &= (1.06 \pm 0.09) \times 10^{-10}, \\
\overline{\mathcal{B}}[B_s \rightarrow \bar{\tau}\tau] &= (7.73 \pm 0.49) \times 10^{-7}, & \overline{\mathcal{B}}[B_d \rightarrow \bar{\tau}\tau] &= (2.22 \pm 0.19) \times 10^{-8},
\end{aligned} \tag{7}$$

with the specified input parameters in that work. The error budget is listed in table 1, showing that currently the largest uncertainties are caused by the imprecise knowledge of CKM parameters and the decay constants. Adding up in quadrature the various uncertainties gives rise to a total relative uncertainty of less than 7 % for $B_s \rightarrow \bar{\ell}\ell$ and below 9 % for $B_d \rightarrow \bar{\ell}\ell$ decays.

Concerning the CKM elements, latest results of 2013 fits from the CKMfitter²⁴ and UTfit (post-EPS13)²⁵ groups have been employed for $|V_{tb}V_{td}^*|$ and the ratio $|V_{tb}V_{ts}^*/V_{cb}|$, where the latter is to very good approximation insensitive to V_{cb} . For the purpose of the predictions of $B_s \rightarrow \bar{\ell}\ell$ modes, the value of $|V_{cb}|_{\text{incl}} = 0.04242 \pm 0.00086$ has been used, which was determined from inclusive semi-leptonic $b \rightarrow c\ell\nu_\ell$ decays²⁶. This value differs by 3.0σ from $|V_{cb}|_{\text{excl}} = 0.03904 \pm 0.00075$ as determined from exclusive $\bar{B} \rightarrow D^*\ell\bar{\nu}_\ell$ decays, based on recent lattice predictions of $B \rightarrow D^*$ form factors²⁷. The usage of the exclusive value of V_{cb} instead of the inclusive would lower the branching ratios of B_s decays by a sizeable amount of $(|V_{cb}|_{\text{excl}}/|V_{cb}|_{\text{incl}})^2 \sim 15\%$, i.e., for example $\overline{\mathcal{B}}[B_s \rightarrow \bar{\mu}\mu] = (3.09 \pm 0.19) \times 10^{-9}$. This result is by 2.4σ lower than the one obtained with the inclusive- V_{cb} value (7), showing the importance of a clarification of the discrepancy of inclusive and exclusive determinations of V_{cb} . In this respect, one might note that the UTfit group provides a global fit where $|V_{cb}|_{\text{excl}} = 0.03955 \pm 0.00088$ (and $|V_{ub}|_{\text{excl}}$) has been used as prior and the fitted posterior value $|V_{cb}| = 0.04121 \pm 0.00050$ was obtained, indicating that other data employed in the global fit prefers also larger values of V_{cb} .

Other uncertainties are below the 2 % level. A number of uncertainties related to higher order corrections are combined in the column labeled “non-parametric” in table 1. They comprise higher order perturbative $\mathcal{O}(\alpha_s^3, \alpha_e^2, \alpha_s\alpha_e)$ matching corrections estimated in^{15,18}, neglected $\mathcal{O}(\alpha_e)$ corrections to (6), the truncated $\text{dim} = 8$ contributions in the OPE of electroweak interactions (2) and others as specified in²³.

In view of the sizeable uncertainties due to CKM elements and decay constants one might consider in the SM the ratio of the branching ratio $B_q \rightarrow \bar{\ell}\ell$ and the mass difference of the

neutral $B_q \bar{B}_q$ system, ΔM_q ,

$$\kappa_{q\ell} \equiv \frac{\bar{\mathcal{B}}[B_q \rightarrow \bar{\ell}\ell] \Gamma_H^q (\Delta M_{B_q})^{-1}}{(G_F m_W m_\ell)^2 \sqrt{1 - 4m_\ell^2/m_{B_q}^2}} \stackrel{\text{SM}}{=} \frac{3|C_{10}(\mu_b)|^2}{4\pi^3 C_{LL}(\mu_b) B_{B_q}(\mu_b)}. \quad (8)$$

In this ratio, both, CKM elements and decay constants, cancel and the only remaining nonperturbative quantity is the so-called bag factor B_{B_q} ²⁰. Here C_{LL} denotes the Wilson coefficient of the SM $\Delta B = 2$ operator (of dim = 6). In the SM the theoretical prediction for the rhs of (8) yields²³

$$\kappa_{s\ell} = 0.0126 \pm 0.0007, \quad \kappa_{d\ell} = 0.0132 \pm 0.0012, \quad (9)$$

which is dominated by the uncertainty of the bag factors, whereas the lhs of (8) together with (1) gives the following experimental values

$$\kappa_{s\ell}|_{\text{exp.}} = 0.0104 \pm 0.0025, \quad \kappa_{d\ell}|_{\text{exp.}} = 0.047 \pm 0.020 \quad (10)$$

that are consistent with the SM predictions. It can be seen that the overall theory uncertainties in $\bar{\mathcal{B}}[B_q \rightarrow \bar{\ell}\ell]$ and $\kappa_{q\ell}$ are quite similar at present.

Alternatively, one might use the experimental measurement of $\Delta M_q|_{\text{exp.}}$ to determine the product $f_{B_q}^2 |V_{tb} V_{tq}^*|^2$ under the assumption of the SM and subsequently predict $\bar{\mathcal{B}}[B_q \rightarrow \bar{\ell}\ell]$, as proposed in²⁸, see²⁹ for a recent update. Recasting (8) into

$$\bar{\mathcal{B}}[B_q \rightarrow \bar{\ell}\ell] = \kappa_{q\ell} \frac{\Delta M_q|_{\text{exp.}}}{\Gamma_H^q} \sqrt{1 - \frac{4m_\ell^2}{m_{B_q}^2}} (G_F m_W m_\ell)^2, \quad (11)$$

it is straightforward to derive the theoretical uncertainties for the branching ratio from the ones of $\kappa_{q\ell}$ and in addition also Γ_H^q , whereas the ones of $\Delta M_q|_{\text{exp.}}$ are numerically negligible at the current level of precision. With the latest experimental numbers $\Delta M_q|_{\text{exp.}}$ and the SM predictions of $\kappa_{q\ell}$ (9) one obtains

$$\bar{\mathcal{B}}[B_s \rightarrow \bar{\mu}\mu] = (3.53 \pm 0.20) \times 10^{-9}, \quad \bar{\mathcal{B}}[B_d \rightarrow \bar{\mu}\mu] = (1.00 \pm 0.09) \times 10^{-10}, \quad (12)$$

which yields in the case of $B_s \rightarrow \bar{\mu}\mu$ values closer to the predictions based on $|V_{cb}|_{\text{incl}}$ than on $|V_{cb}|_{\text{excl.}}$

4 Conclusions

The latest standard model predictions have been presented for the branching ratios of rare leptonic $B_q \rightarrow \bar{\ell}\ell$ ($q = d, s$ and $\ell = e, \mu, \tau$) decays. The inclusion of NLO electroweak and NNLO QCD corrections decrease previous renormalization scheme dependences from about 7 % to 0.6 % and 1.8 % to 0.2 %, respectively. The current uncertainties of the branching ratios are below 7 % for $B_s \rightarrow \bar{\ell}\ell$ and 9 % for $B_d \rightarrow \bar{\ell}\ell$ channels, where the dominant source are the imprecise knowledge of the CKM matrix elements V_{cb} and V_{td} and comparable uncertainties from the decay constants of B_q mesons. In this respect, the ratios $\kappa_{q\ell} \propto \bar{\mathcal{B}}[B_q \rightarrow \bar{\ell}\ell]/\Delta M_q$ are free of the decay constant and CKM factors in the SM. Their current predictions are dominated by the uncertainties due to the bag factors in $\Delta B = 2$ hadronic matrix elements and have comparable size to the ones of the branching ratios.

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